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Optimum heating of thick plate

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ABSTRACT

Five different methods for determining optimum fluid temperature changes during heating or cooling thick plate are examined. The first method presented in this paper is based on the discrete form of Duhamel's integral. The method can be easily applied to bodies of complex shapes and is recommended for use, while three other methods: the Laplace transform, Burggraf analytical method, and space marching method can be applied to only a very limited number of optimum control problems. In the fifth approach, the optimum fluid temperature changes are approximated by a function with unknown parameters, which are determined using the least squares method. These five techniques were used to determine time changes of fluid temperature assuring linear increase of the slab wall temperature at the given location inside the body. No one approach is perfect. The optimum fluid temperature changes are burdened with a large uncertainty at the beginning of the heating process.

1. Introduction

Large thermal stresses can occur at the inner surface of thick walled pressure components of steam boilers during the start-up and shut-down operations [1,2]. Measurements of strains or stresses on the inner component surface, which is exposed to hot fluid under high pressure, is extremely difficult. For this reason, the thermal stresses at the inner surface are monitored indirectly by measurements of time variations of the component wall temperature at the interior location or at the outer thermally insulated surface, which are easily accessible. From the solution of the inverse heat conduction problem (IHCP) the spatial temperature distribution in the whole component for any time is determined. In computer monitoring systems used in power plants, the time and space temperature and stress distribution in the pressure components is calculated sequentially in an on-line mode in order to inform the operator or the control system to take measures to speed up or slow down the start-up or shut-down process.

Determination of optimum fluid temperature changes is also an inverse heat conduction problem. To avoid excess thermal stresses, the temperature of the component wall should be increased or decreased according to the prescribed function of time. Another option is to adjust the fluid temperature changes in such a way that the thermal stress at the point of stress concentration does not exceed the allowable values [3–6]. With the exception of the earliest time period, the optimum rate of fluid and wall temperature changes is constant [3,5,6], if the physical properties of the component material and allowable stress are constant. This optimum rate of fluid temperature changes can easily be determined based on the quasi-steady state theory [5]. However, determining fluid temperature changes at the earliest time of the transient process, which assure that the calculated temperature at the outer component surface is equal to the measured values or changes according to the prescribed time function, is a very difficult task.

Many numerical [7–9], analytical [10–13], and semi-analytical [14] approaches have been developed for solving IHCPs. Explicit analytical solutions are limited to simple geometries, but are very efficient computationally and are of fundamental importance for investigating basic properties of IHCPs.

The problem of optimum heating or cooling will be solved under the assumption that physical properties of the component material and the heat transfer coefficient are constant.

The aim of the paper is to show how difficult the IHCP is. Even for the simple IHCP, when the temperature changes are exactly known at the interior point, it is impossible to find a unique solution at the initial stage of the transient process. The discussion focuses on sequential inverse methods, which are widely used in on-line thermal stress monitoring systems [5]. Whole domain estimation procedures, based on simultaneously determining all the unknown parameters for the total time interval, are less appropriate for on-line applications, since the entire time history of the measured temperature is not known in advance. Data points are only available over the time interval from initial time till the moment under consideration. In addition, the time of computation should be smaller than the data sampling period.

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Nomenclature

Bi	Biot number, $Bi=hL/k$
c	specific heat capacity
d	distance from the point at which the wall temperature is prescribed to the exposed surface of the slab
F	number of future time steps
F_1, F_2	functions defined by Eqs. (35) and (36)
Fo	Fourier number, $Fo=\alpha t/L^2$
h	heat transfer coefficient
k	thermal conductivity
L	slab thickness
n_t	number of time points
$\dot{q}$	heat flux density
$\mathbf{r}$	position vector
S	solution of the initial-boundary problem
S_L	sum of the temperature difference squares
s	complex variable
T	temperature
t	time
u	influence function (temperature of the slab for unit step-wise increase of the fluid temperature)
v_T	rate of fluid temperature changes
x	cartesian coordinate
x_T	coordinate of the point at which time change of the slab temperature is prescribed
y	temperature prescribed at the location x_T

Greek symbols

α	thermal diffusivity
Δt	time step
μ_n	root of the characteristic equation
ρ	density

Subscripts

f	fluid
0	initial

A new procedure for the solution of the linear IHCP based on a numerical approximation of Duhamel's integral in conjunction with future time steps is presented.

This method and four other techniques: the Laplace transform method, the modified Burggraf solution, the space marching method, and the whole domain least squares method will be used for solving optimum heating problem when linear time temperature variation is prescribed at the insulated rear surface of the plate. New solutions for the fluid temperature changes for this specific problem will be found.

Moreover, the optimum fluid temperature changes obtained by the various methods considered in the paper will be compared.

It must be emphasized that all the methods analyzed in the paper exhibit some problems in determining initial optimum fluid temperature changes if the wall temperature changes are prescribed at the rear insulated surface. The optimum fluid temperature changes, obtained by various methods, differ significantly. The problems encountered in the methods used in the paper will be discussed in detail.

2. Mathematical formulation of the problem

In the case of time-dependent boundary conditions, the solution for the linear initial-boundary problem can be significantly simplified by applying Duhamel's integral

$$S(\mathbf{r}_p,t) = S_0 + \int_0^t [T_f(\Theta) - T_0]\frac{\partial u(\mathbf{r}_P, t-\Theta)}{\partial t} d\Theta = S_0 + \int_0^t \frac{d[T_f(\Theta)-T_0]}{d\Theta} u(\mathbf{r}_P, t-\Theta) d\Theta, \quad (1)$$

where $S(\mathbf{r}_P, t)$ is the solution of the initial-boundary problem with time-dependent fluid temperature $T_f(t)$ at the location $\mathbf{r}_p$ and time t. The initial value S_0 is a constant and does not depend on the location. Function $u(\mathbf{r}_P, t)$ is the solution for the initial-boundary problem with unit step increase of the fluid temperature $T_f(t) = 1$, $t > 0$. When the solution $u(\mathbf{r}_P, t)$ for the unit fluid temperature change described by the Heaviside function is known, it is easy to determine the solution $S(\mathbf{r}, t)$ for the time-dependent fluid temperature $T_f(t)$. In optimization of heating or cooling of the construction element, the desired system response $y(t) = S(\mathbf{r}_P, t)$at the inner point $\mathbf{r}_P$ is given and the time changes of the fluid temperature $T_f(t)$

$$\int_0^t [T_f(\Theta) - T_0]\frac{\partial u(\mathbf{r}_P, t-\Theta)}{\partial t} d\Theta = y(t) - S_0 \quad (2)$$

is searched for. The optimization problem is reduced to the solution of the Volterra integral equation of the first kind.

To evaluate the convolution integral in Eq. (2), the real changes of the function $f(\Theta) = T_f(\Theta) - T_0$ are replaced by a step-wise function

$$\begin{aligned} f_1 &= f(\Theta_1/2), \quad 0 \leqslant \Theta \leqslant \Theta_1 \\ f_2 &= f[\Theta_1 + (\Theta_2 - \Theta_1)/2] \quad \Theta_1 \leqslant \Theta \leqslant \Theta_2 \\ &\vdots \\ f_M &= f[\Theta_{M-1} + (\Theta_M - \Theta_{M-1})/2] \quad \Theta_{M-1} \leqslant \Theta \leqslant \Theta_M \end{aligned} \quad (3)$$

where M is the number of time steps.

A simple way of determining the integral in Eq. (2) is the method of rectangles. However, in case of too small integration time steps $\Delta\Theta_i = \Theta_i - \Theta_{i-1}$ unexpected instabilities can appear in the estimated function $f(t)$.

To assure the stability of the calculations, the time step Δt of determining the fluid temperature $T_f(t)$ should be larger than the critical value Δt_{cr} evaluated from the following condition:

$$\Delta t_{cr} = \frac{(\Delta Fo_{cr})d^2}{\alpha} \quad (4)$$

where $\Delta Fo_{cr} \cong 0.5$ and $d = \min|\mathbf{r}_P\text{-}\mathbf{r}_S|$ is the distance between the point P and the body surface and α denotes the thermal diffusivity of the body. In practice, time step Δt given by Eq. (4) is too big, making it impossible to reconstruct the rapidly changing fluid temperature $T_f(t)$ that causes a specified output $y(t)$ inside the body or on its surface.

This problem is especially important at the beginning of the optimum heating or cooling process. In this paper, so-called future time steps are used for stabilising the solution of the inverse problem. These were introduced by Beck et al. [7] in the inverse problem analysis. The efficiency of the future time steps results from the artificial extension of the basic time step Δt. This approach is very useful in inverse problems because the time changes of the fluid temperature $T_f(t)$ received in the interior point $\mathbf{r}_p$ are significantly delayed and damped.

Evaluation of f_i, $i = 1,\ldots,M$ will be performed step by step. First, f_1 will be evaluated, then f_2, etc. In each case it is assumed that the values of $S(\mathbf{r}_P, t_{M-1})$ and f_{M-1} are known. The value f_M remains to be

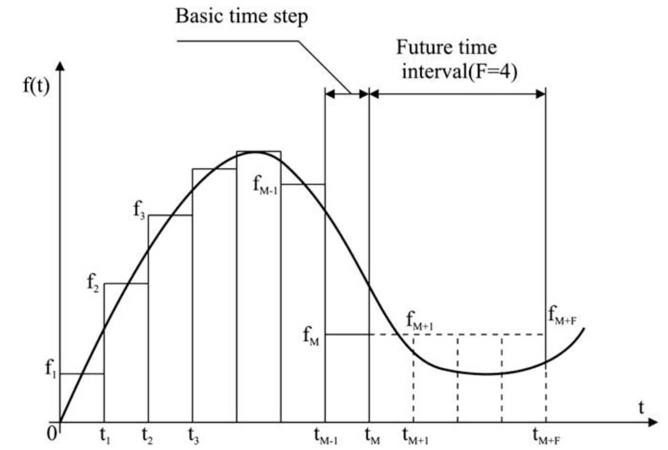

Fig. 1. Step-wise approximation for $f(t) = T_f(t) - T_0$ and use of future temperatures to stabilise the inverse problem of optimum heating.

calculated in the time interval $t_{M-1} \leqslant t \leqslant t_M$ (Fig. 1). The time interval will be artificially extended by F future time steps, with the assumption, that in the extended interval, the $t_{M-1} \leqslant t \leqslant t_{M+F}$ value of the function $f(t)$ remains constant and equals f_M, i.e.,

$$f_{M+1} = f_{M+2} = \cdots = f_{M+F} = f_M \tag{5}$$

The value of the fluid temperature (input signal) $f_M = T_f\ (t_M)$ is determined from Eq. (2) which can be written in the following form:

$$S(\mathbf{r}_P, t_{M+F}) - y(t_{M+F}) = 0 \tag{6}$$

It is assumed that the value f_M evaluated in this way is valid only in the interval $t_{M-1} \leqslant t \leqslant t_M$. The convolution integral in Eq. (1) can be calculated numerically using the method of rectangles

$$\begin{aligned} S(\mathbf{r}_P, t_{M+F}) &= S_0 + \int_0^{t_{M+F}} f(\Theta) \frac{\partial u(\mathbf{r}_P, t-\Theta)}{\partial \Theta} d\Theta \\ &\cong S_0 + f_1 \Delta u_{M+F} + f_2 \Delta u_{M+F-1} + \cdots + f_{M+F} u_1 \\ &= S_o + \sum_{i=1}^{M-1} f_i \Delta u_{M+F-i} + (\Delta u_F + \Delta u_{F-1} + \cdots + u_1) f_M \\ &= S_0 + \sum_{i=1}^{M-1} f_i \Delta u_{M+F-i} + u_{F+1} f_M, \end{aligned} \tag{7}$$

where

$$\Delta u_0 = u_1 - u_0 = u_1, \quad \Delta u_i = u_{i+1} - u_i, \quad u_n = \sum_{i=0}^{n-1} \Delta u_i. \tag{8}$$

Substituting (7) into (6) yields

$$f_1 = \frac{y(t_{F+1}) - S_0}{u_{F+1}} \tag{9}$$

and

$$f_M = \frac{y(t_{M+F}) - S_0 - \sum_{i=1}^{M-1} f_i \Delta u_{M+F-i}}{u_{F+1}}, \quad M = 2, 3, \ldots \tag{10}$$

Eqs. (9) and (10) allow us to sequentially calculate f_1, f_2, f_3, etc. The time interval $\Delta t_M = t_M - t_{M-1}$ does not have to be as large as given by Eq. (4). If $F \geqslant 1$, the time step Δt can be several times smaller. Stable solutions are already obtained at $\Delta Fo_{cr} \cong 0.05$. In comparison with $\Delta Fo_{cr} \cong 0.5$, $\Delta Fo_{cr} \cong 0.05$ denotes a significant increase in the frequency of determining f_i, $i = 1, 2, 3, \ldots$. Future time steps increase the stability of the calculations, however they diminish the accuracy of $f(t)$ estimation.

The Laplace transform is another method that allows determining optimum fluid temperature changes. By taking the Laplace transforms of both sides of Eq. (2) we obtain

$$\left[\overline{T}_f(s) - \frac{T_0}{s}\right] [s\bar{u}(\mathbf{r}_P, s) - u(\mathbf{r}_P, 0)] = \bar{y}(s) - \frac{S_0}{s} \tag{11}$$

which can be solved for $\overline{T}_f(s)$, so that

$$\overline{T}_f(s) = \frac{T_0}{s} + \frac{\bar{y}(s) - \frac{S_0}{s}}{s\bar{u}(\mathbf{r}_P, s) - u(\mathbf{r}_P, 0)}, \tag{12}$$

where s is the complex variable. The initial value of $u(\mathbf{r}_P, t)$ is $u(\mathbf{r}_P, 0) = 0$. In order to find the corresponding function of time $T_f(t)$, an inversion of the Laplace transform (12) can be performed analytically or numerically. In the numerical procedures for finding the inverse Laplace transforms, the time sampling interval Δt must be sufficiently large enough to obtain stable results for $T_f(t)$. This is the main drawback of using the Laplace transform based method to determine the optimum changes of $T_f(t)$. If the point $\mathbf{r}_p$ is situated at a large distance from the heated surface, then the time interval Δt should be large enough to assure the stability of the solution. The requirement of large time steps does not allow determining quick changes of the fluid temperature $T_f(t)$, which occur at the initial period of optimum heating.

The third method, which can be used to determine the optimum fluid temperature $T_f(t)$, is the analytical solution of the inverse heat conduction problem presented by Burggraf in [15]. The Burggraf method is appropriate only for one-dimensional problems and does not allow us to determine the optimum temperature $T_f(t)$ at the initial time period.

The same drawback has the solution obtained by the space marching method [5,14], which is identical to that obtained by the Burggraf method.

In the fifth method a time function representing optimum changes of the fluid temperature is assumed and unknown parameters are estimated using the least squares method.

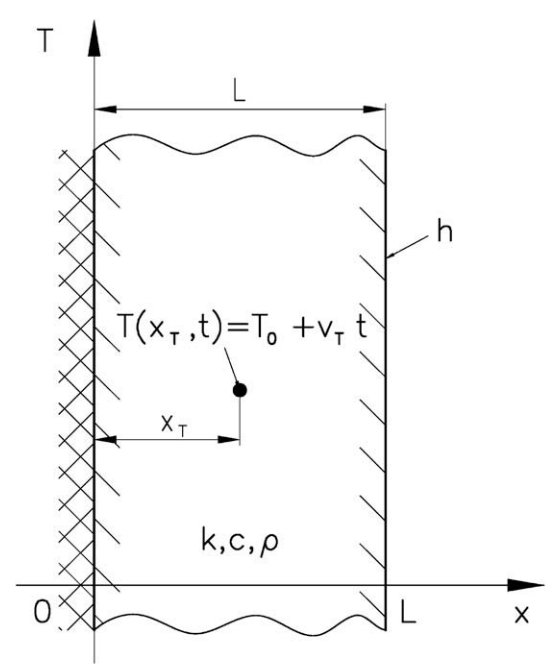

Fig. 2. Location x_T, at which temperature changes according to the prescribed function $T(x_T, t) = T_0 + v_T t$.

3. Example of applications

The objective of this example is to determine the optimum temperature changes $T_f(t)$ for which the temperature of the slab in the point x_T increases linearly with time

$$S(\mathbf{r}_P, t) = T(x_T, t) = v_T t, \tag{13}$$

where v_T is the constant temperature rate (Fig. 2). The mathematical formulation of the problem is

$$c\rho \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \quad t > 0 \tag{14}$$

$$T(x, 0) = T_0, \quad 0 \leqslant x \leqslant L \tag{15}$$

$$\frac{\partial T}{\partial x}\Big|_{x=0} = 0, \quad t > 0 \tag{16}$$

$$T(x_T, t) = y(t) = T_0 + v_T \cdot t \qquad 0 \leqslant x_T \leqslant L, \quad t > 0 \tag{17}$$

$$k\frac{\partial T}{\partial x}\Big|_{x=L} = h[T_f(t) - T|_{x=L}], \quad t > 0 \tag{18}$$

The boundary condition of the third kind is given at the exposed surface. The time varying fluid temperature $T_f(t)$ will be determined from the solution of the problem (14)–(18).

Calculations were carried out for the slab of thickness $L = 0.1$ m with the uniform initial temperature $T_0 = 0$ °C. The physical properties of the slab made from carbon steel are: $\rho = 7800$ kg/m^3, $c = 482$ J/kg K, $k = 42$ W/m K, where ρ denotes the density, c is the specific heat capacity and k is the thermal conductivity. The heat transfer coefficient at the exposed slab surface is $h = 2000$ W/m^2 K. The rear surface of the slab is thermally insulated. The temperature of the rear surface ($x_T = 0$) increases with the constant rate $v_T = 0.1$ K/s. The methods described above will be used for determining $T_f(t)$.

3.1. Approximate solution based on the numerical integration of the convolution integral

In order to calculate f_i, $i = 1, \ldots, M$ by using formulas (9) and (10) it is necessary to solve the heat conduction equation (14) with the conditions (15), (16), and (18) for $T_f = 1$ °C, $t > 0$. The exact analytical solution of this problem is [16]:

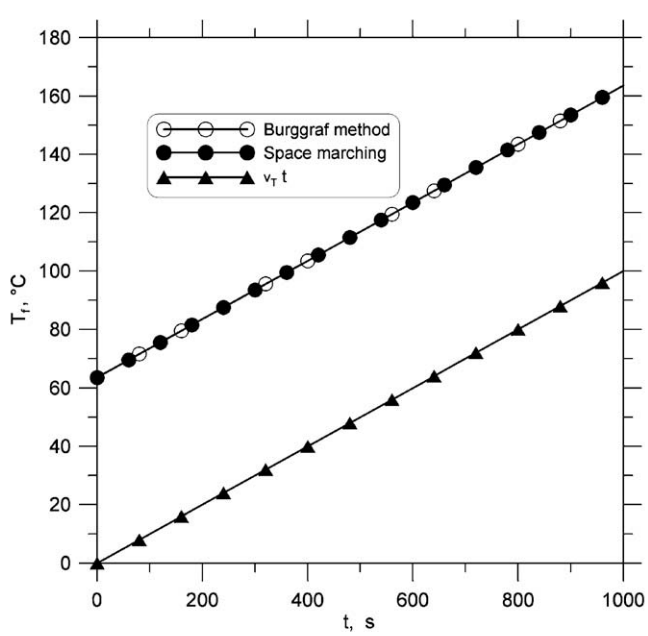

Fig. 3. Optimum fluid temperature changes obtained by using the Burggraf solution and space marching method.

$$u(x, t) = 1 - \sum_{n=1}^{\infty} \frac{2\sin\mu_n}{\mu_n + \sin\mu_n \cos\mu_n} \cos\left(\mu_n \frac{x}{L}\right) \exp\left(-\mu_n^2 \frac{\alpha t}{L^2}\right), \tag{19}$$

where $\alpha = k/c\rho$ is thermal diffusivity, μ_n are the positive roots of $\mu_n tg\mu_n = Bi$, and $Bi = hL/k$ is the Biot number.

Optimum fluid temperature changes are shown in Fig. 3. Since the point P is located on the rear, insulated surface not directly adjacent to the fluid, the future time steps were used ($F = 5$). The phenomenon of strong damping and delaying of temperature changes $T_f(t)$ at the body inside causes large discrepancies in the estimated changes of $T_f(t)$ for small time values. After the initial allowable temperature increase over 200 °C, a rapid decrease in temperature T_f to about 80 °C occurs, and then the temperature rises with a constant rate of $v_T = 0.1$ K/s.

3.2. Determination of the optimum fluid temperature changes using the Laplace transform

If the prescribed temperature changes of the slab at location x_T are given by Eq. (13), then the Laplace transform (12) becomes

$$\overline{T}_f(s) = \frac{T_0}{s} + \frac{v_T}{s^2} \frac{\frac{k}{h} q \sinh(qL) + \cosh(qL)}{s^2 \cosh(q x_T)}, \tag{20}$$

where $q = \sqrt{s/\alpha}$.

Applying the inverse Laplace transforms to Eq. (20) gives

$$\begin{aligned} T_f(t) = {} & T_0 + v_T\left\{t + \frac{L^2}{\alpha}\left[\frac{1}{Bi} + \frac{1}{2}\left(1 - \frac{x_T^2}{L^2}\right)\right]\right\} \\ & + \frac{2x_T^2 v_T}{\alpha} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n-1}{2}\pi \frac{L}{x_T}\right) - \frac{1}{Bi}\left(\frac{L}{x_T}\right)\left(\frac{2n-1}{2}\pi\right) sin\left(\frac{2n-1}{2}\pi \frac{L}{x_T}\right)}{(-1)^{n-1}\left(\frac{2n-1}{2}\pi\right)^3} \\ & \times \exp\left[-\frac{\alpha t}{x_T^2}\left(\frac{2n-1}{2}\pi\right)^2\right] \end{aligned} \tag{21}$$

where $Bi = hL/k$. The expression (21) represents the exact solution of the problem (14)–(18). In practice, it is difficult to find temperature $T_f(t)$ for $0 < (x_T/L) \ll 1$, because the exponential term in (21) is close to zero and Eq. (21) reduces to the quasi-steady-state solution

$$T_f(t) = T_0 + v_T\left\{t + \frac{L^2}{\alpha}\left[\frac{1}{Bi} + \frac{1}{2}\left(1 - \frac{x_T^2}{L^2}\right)\right]\right\}. \tag{22}$$

Thus, the form of Eq. (21) is not appropriate for determining optimum temperature changes of the fluid in the early time stages of the slab heating, if $x_T/L \ll 1$.

3.3. Determination of the optimum fluid temperature changes by using the Burggraf and space marching methods

Burggraf presented one of the earliest analytical solutions of the one-dimensional inverse heat conduction problem [15]. When $x_T = 0$, then applying the Burggraf method to the inverse problem (14)–(17) yields

$$T_B(x, t) = y(t) + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{x^{2n}}{\alpha^n} \frac{d^n y}{dt^n} = T_0 + v_T t + v_T \frac{x^2}{2\alpha}. \tag{23}$$

Inserting Eq. (23) in (18) and solving for $T_f(t)$ gives

$$T_f(t) = T_0 + v_T\left[t + \frac{L^2}{\alpha}\left(\frac{1}{Bi} + \frac{1}{2}\right)\right]. \tag{24}$$

Thus, the obtained result (24) is identical with the quasi-steady-state solution (22) and is not adequate for small time values. It is worth mentioning that the same solution (24) gives the space marching method [5,14].

Examining the function (23) at $t = 0$ shows, that the initial temperature distribution is non-uniform

$$T_B(x,0) = T_0 + \frac{v_T x^2}{2\alpha}. \tag{25}$$

In order to alleviate this difficulty the general problem given by Eqs. (14)–(17) may be separated in accordance with the superposition method into a set of simpler problems containing: a homogenous transient problem and quasi-steady-state problem [5]. The optimum fluid temperature changes $T_f(t)$, which were calculated using Eq. (24) and the space marching method developed in [14], are shown in Fig. 3. Taking into account that the temperatures at the boundaries are: $T_B(0, t) = v_T t$ and $T_B(L, t) = v_T[t + L^2/(2\alpha)]$, the method of superposition will be used to satisfy the initial condition given by Eq. (15). The transient temperature distribution T_T is the solution of the heat conduction equation:

$$c\rho\frac{\partial T_T}{\partial t} = k\frac{\partial^2 T_T}{\partial x^2}, \quad t > 0, \tag{26}$$

with the boundary conditions

$$T_T|_{x=0} = 0, \tag{27}$$

$$T_T|_{x=L} = 0, \tag{28}$$

and the initial condition

$$T_T|_{t=0} = -\frac{v_T x^2}{2\alpha}. \tag{29}$$

Solving the initial-boundary value problem defined by Eqs (26)–(29) using the method of variable separation yields the transient temperature distribution $T_T(x, t)$ [17]. The complete solution for temperature distribution$T = T_B + T_T$, that satisfies the initial condition (15), can be expressed as

$$T(x,t) = T_0 + v_T\left(t + \frac{x^2}{2\alpha}\right) + \frac{v_T L^2}{\alpha}\sum_{n=1}^{\infty}\frac{1}{n\pi}\left[(-1)^n\left(1 - \frac{2}{n^2\pi^2}\right) + \frac{2}{n^2\pi^2}\right] \times \sin\left(n\pi\frac{x}{L}\right)\exp\left(-n^2\pi^2\frac{\alpha t}{L^2}\right). \tag{30}$$

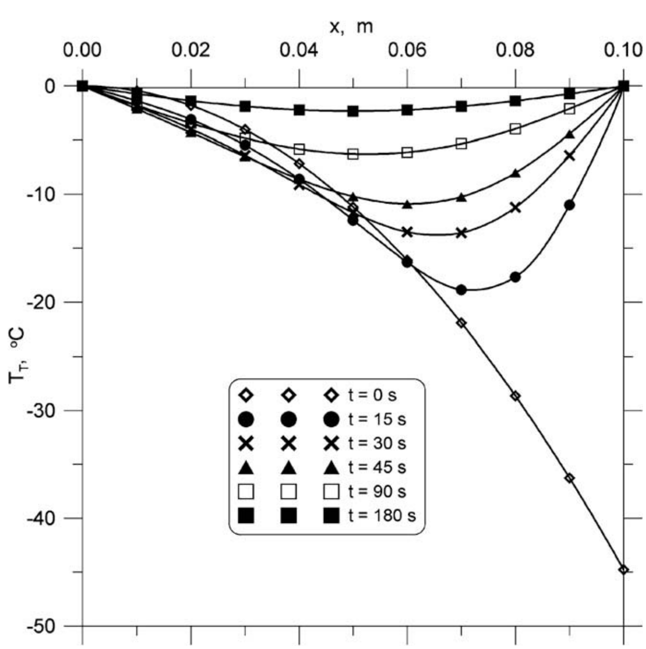

Fig. 4. Transient part of temperature distribution which is needed for modifying the Burggraf solution.

The transient part T_T of the complete solution T which is needed to improve the quasi-steady-state solution (Burggraf solution) is shown in Fig. 4.

The heat flux $\dot{q}(x, t)$ is given by

$$\dot{q}(x,t) = k\frac{\partial T}{\partial x} = c\rho\, v_T x + c\rho\, v_T L \times \sum_{n=1}^{\infty}\left[(-1)^n\left(1 - \frac{2}{n^2\pi^2}\right) + \frac{2}{n^2\pi^2}\right] \times \cos\left(n\pi\frac{x}{L}\right)\exp\left(-n^2\pi^2\frac{\alpha t}{L^2}\right). \tag{31}$$

The optimum heat flux at the exposed surface is shown in Fig. 5.

The optimum fluid temperature $T_f(t)$ is determined from Eq. (18) as

$$T_f(t) = T|_{x=L} + \frac{k}{h}\frac{\partial T}{\partial x}\bigg|_{x=L}, \tag{32}$$

where $T(x, t)$ and $k\,\partial T/\partial x$ are defined by Eqs. (30) and (31), respectively. The optimum temperature history $T_f(t)$ calculated according to Eq. (32) is compared in Fig. 6 with the approximate solution based on the numerical integration of the convolution integral. The deviations between the estimated and optimum temperatures $T_f(t)$ were found to decrease with time. The temperature distribution across the slab thickness during the optimum heating is shown in Fig. 7.

3.4. Determination of the optimum fluid temperature changes by solving the parametric least squares problem

Optimum changes of the fluid temperature T_f during heating of the slab, which are shown in Fig. 6, are very difficult to carry out in practice for the initial stage of the component heating. However, optimum fluid temperature changes can be approximated by a ramp function consisting of a step increase in fluid temperature T_s followed by the temperature increase with a constant rate v_T. The solution of the direct heat conduction problem, which is defined by the heat conduction equation (14), initial condition (15), boundary conditions (16) and (18) with the fluid temperature given by

$$T_f(t) = T_0 + T_s + v_T t, \tag{33}$$

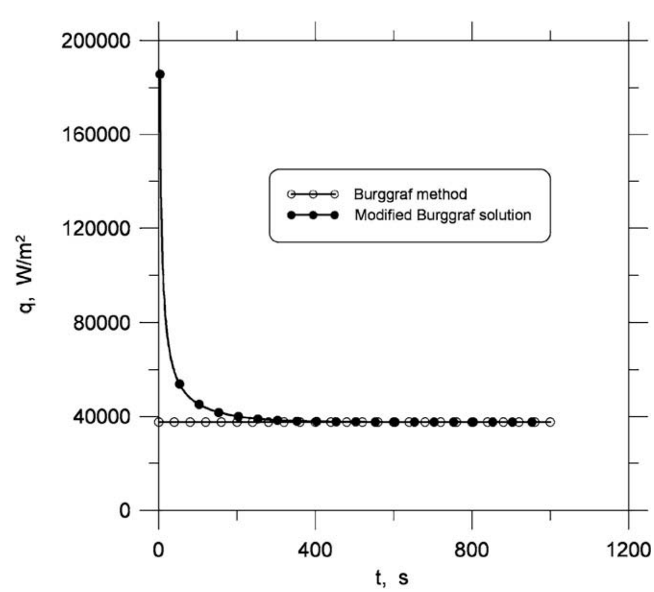

Fig. 5. Heat flux at the exposed surface during optimum heating.

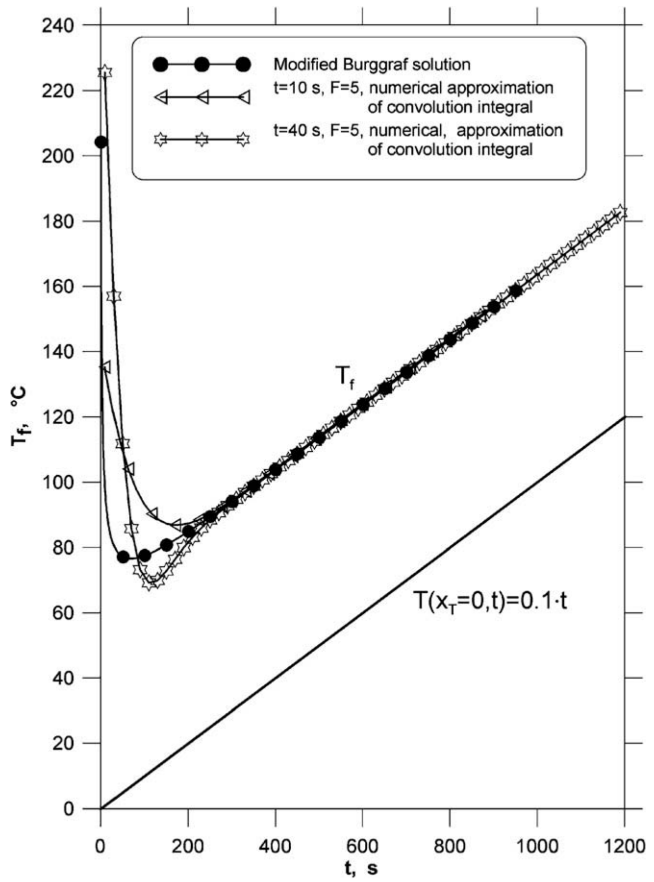

Fig. 6. Optimum changes of the fluid temperature T_f during heating of the slab with the prescribed temperature of the insulated rear side: $T(0, t) = 0.1t$.

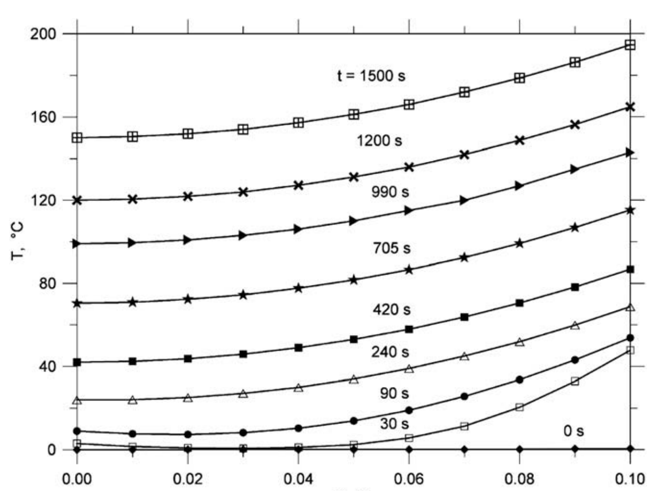

Fig. 7. Transient temperature distribution in a slab during optimum heating.

is as follows

$$T(x,t) = T_0 + \int_0^t \frac{d[T_f(\theta) - T_0]}{dt} u(x, t-\theta) d\theta, \tag{34}$$

where the influence function $u(x,t)$ is defined by Eq. (19).

Substituting Eqs. (19) and (33) into Eq. (34) and integrating gives

$$T(x,t) = T_s + (T_0 - T_s)F_1(x,t) + v_T t - v_T F_2(x,t), \tag{35}$$

where

$$F_1(x,t) = \sum_{n=1}^{\infty} \frac{2\sin\mu_n}{\mu_n + \sin\mu_n \cos\mu_n} \cos\left(\mu_n \frac{x}{L}\right) \exp\left(-\mu_n^2 \frac{\alpha t}{L^2}\right), \tag{36}$$

$$F_2(x,t) = \frac{L^2}{\alpha}\left[\frac{1}{Bi} + \left(1 - \frac{x}{L}\right) - \frac{1}{2}\left(1 - \frac{x}{L}\right)^2 - \sum_{n=1}^{\infty} \frac{2\sin\mu_n}{\mu_n^2(\mu_n + \sin\mu_n \cos\mu_n)} \cos\left(\mu_n \frac{x}{L}\right) \exp\left(-\mu_n^2 \frac{\alpha t}{L^2}\right)\right] \tag{37}$$

In the inverse problem, the unknown parameters T_s and v_T are to be adjust to satisfy approximately the following system of equations:

$$y(t_i) - T(x_T, t_i) \cong 0, \quad i = 1, \ldots, n_t. \tag{38}$$

where the prescribed temperature $y(t)$ changes at the location x_T are given by Eq. (17).

The least squares method is used to estimate parameters T_s and v_T. The parameters T_s and v_T are computed by minimizing the sum of squares of the differences between values given by the model (35) and those obtained from Eq. (17):

$$S_L = \sum_{i=1}^{n_t} [y(t_i) - T(x_T, t_i)]^2. \tag{39}$$

It is necessary to find the values of T_s and v_T, for which the two partial derivatives are simultaneously zero:

$$\frac{\partial S_L}{\partial T_s} = 0, \quad \frac{\partial S_L}{\partial v_T} = 0. \tag{40}$$

Finding derivatives (40) gives a set of linear equations in the unknowns T_s and v_T, which has the following solution:

$$T_s = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, \tag{41}$$

$$v_T = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}}, \tag{42}$$

where

$$a_{11} = \sum_{i=1}^{n_t} [1 - F_1(x_T, t_i)]^2,$$

$$a_{12} = \sum_{i=1}^{n_t} [1 - F_1(x_T, t_i)][t_i - F_2(x_T, t_i)], \quad a_{21} = a_{12},$$

$$a_{22} = \sum_{i=1}^{n_t} [t_i - F_2(x_T, t_i)]^2,$$

$$b_1 = -\sum_{i=1}^{n_t} [1 - F_1(x_T, t_i)][T_0 F_1(x_T, t_i) - y_i],$$

$$b_2 = -\sum_{i=1}^{n_t} [t_i - F_2(x_T, t_i)][T_0 F_1(x_T, t_i) - y_i]. \tag{43}$$

The discrepancy between the prescribed function $y(t)$ and the fitting function $T(x_T, t)$ can be quantified by the mean square error (standard deviation), defined as

$$\sigma = \sqrt{\frac{S_L}{n_t - m}}. \tag{44}$$

The symbol m denotes the number of parameters to be estimated. In this example, m is 2. The values of T_s and v_T are also determined by the modified Levenberg–Marquardt method [18,19] using the subroutine BCLSF from the IMSL mathematical library [20].

As an example, we shall fit the solution (35) to the function $y = 0.1t$. As the time points t_i are equally distributed with the time step Δt, the time points t_i are given by

$$t_i = i\Delta t, \quad i = 1, \ldots, n_t. \tag{45}$$

The number of time points n_t is 100. Parameters T_s and v_T were computed for three different time steps Δt: 12 s, 30 s, and 60 s. The following results were obtained:

T_s = 81.2176 °C, v_T = 0.07308 K/s, S_L = 984.73 K^2, σ = 3.17 K for Δt = 12 s,
T_s = 68.8978 °C, v_T = 0.09700 K/s, S_L = 706.43 K^2, σ = 2.68 K for Δt = 30 s,
T_s = 65.8331 °C, v_T = 0.09939 K/s, S_L = 440.35 K^2, σ = 2.12 K for Δt = 60 s.

The same results were obtained using the Levenberg–Marquardt method.

The computed fluid and wall temperatures are shown in Fig. 8(a)–(c).

4. Conclusions

Five different methods for predicting optimum temperature changes of the heating fluid were presented. The solutions based on the numerical approximation of the convolution integral compare favourably with other methods and can be used to determine the optimum fluid temperature with respect to the prescribed temperature or stress histories at the interior location. This method is appropriate for bodies with complex shapes. The influence function can be computed using the finite element method.

The solution for the optimum temperature of the fluid obtained by using the Laplace transform method is accurate, if the desired temperature history is prescribed at locations near the heated surface. The Burggraf method is inaccurate to predict optimum temperature changes in the early stages of heating. It requires an improvement of the inverse problem solution to account for the non-uniform temperature distribution resulting from the quasi-

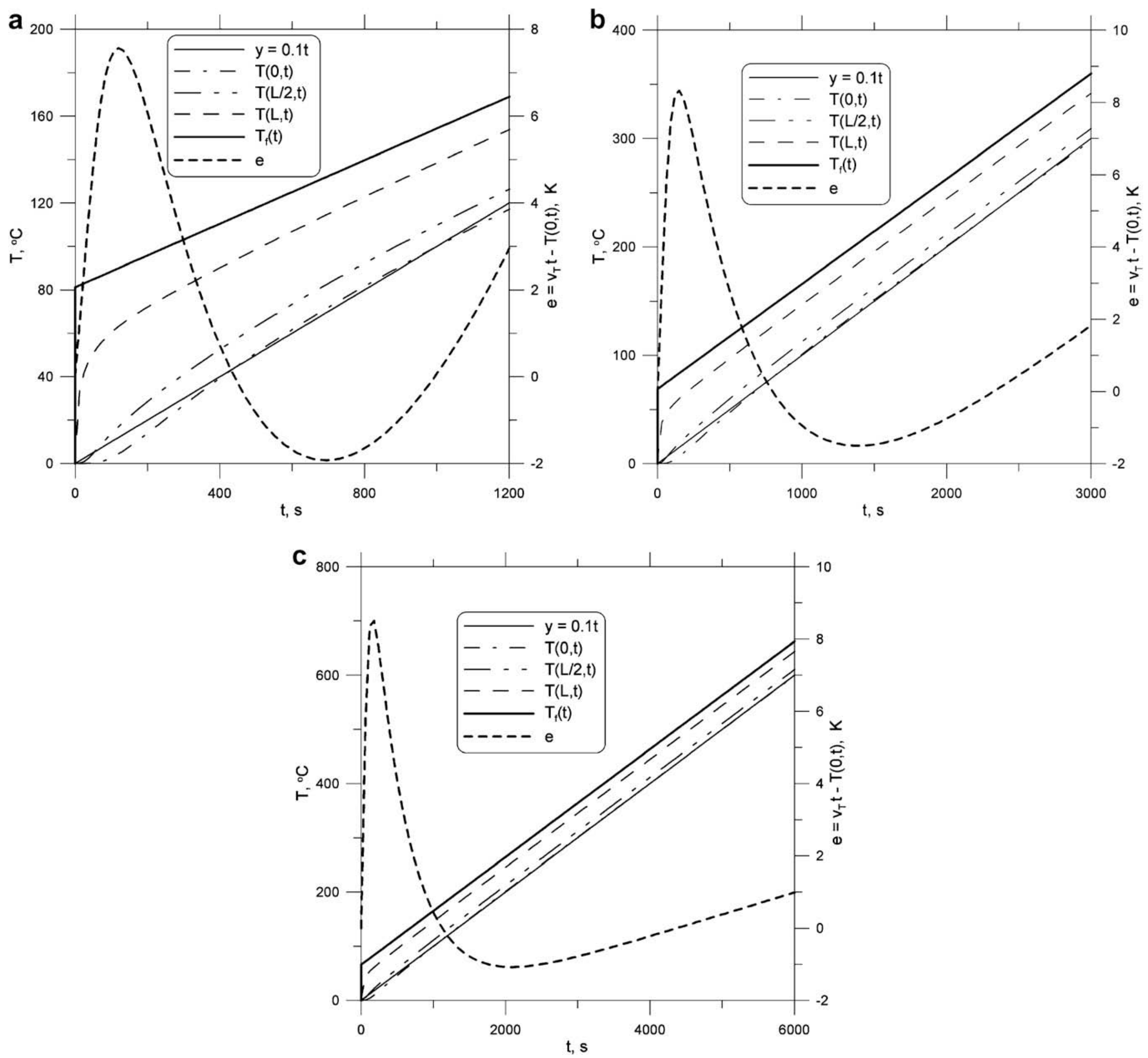

Fig. 8. Fluid and slab temperature changes when optimum fluid temperature changes are approximated by a ramp function: (a) Δt = 12 s, (b) Δt = 30 s, and (c) Δt = 60 s.

steady-state temperature distribution. The identical results to those obtained by the Burggraf method gives the space marching method. In the fifth method the optimum fluid temperature is approximated by an appropriate function of time and unknown parameters.

All the analyzed methods are not able to find an exact or accurate optimum fluid temperature changes at the beginning of the heating process.

The difficulties encountered in searching of the solution of the transient IHCP realize us, that we can find very often only very approximate solution.

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